Number Theory and the Circle Packings of Apollonius

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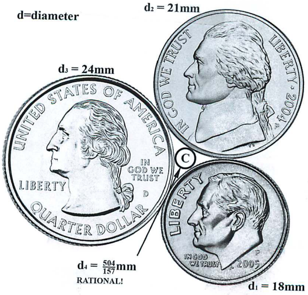
Apollonius of Perga
lived from about 262 BC to about 190 BC
Apollonius was known as ‘The Great Geometer’. His famous book *Conics* introduced the terms parabola, ellipse, and hyperbola.


d=diameter
$d_2 = 21$mm
$d_3 = 24$mm
C
$d_4 = \frac{504}{157}$mm
RATIONAL!
$d_1 = 18$mm

Scale the picture by a factor of 252 and let
$a(c)$ = curvature of the circle $c = 1/\mathrm{radius}(c)$.

The curvatures are displayed. Note the outer one by convention has a negative sign.
By a theorem of Apollonius, place unique circles in the lines.

The Diophantine miracle is the curvatures are integers!

Repeat ad infnitum to get an integral Apollonian packing:

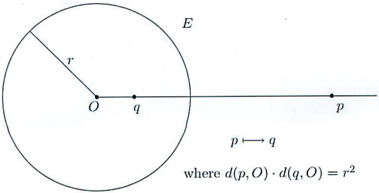
- There are infinitely many such P's.

Basic questions (Diophantine)
Which integers appear as curvatures?
Are there infinitely many prime curvatures, twin primes i.e. pairs of tangent circles with prime curvature?

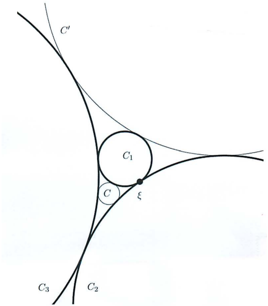
- The integral structure — F. Soddy (1936)
- Diophantine setup and questions — R. Graham–J. Lagarias–C. Mellows–L. Wilks–C. Yan (2000)
- Many of the problems are now solved
  Recent advances in modular forms, ergodic theory, hyperbolic geometry, and additive combinatorics.

Apollonius' Theorem

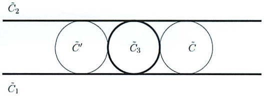
Given three mutually tangent circles c_1, c_2, c_3, there are exactly two circles c and c' tangent to all three.



Inversion in a circle takes circles to circles and preserves tangencies and angles.



c_1, c_2, c_3 given invert in ξ ($\xi \to \infty$) yields



Now the required unique circles $\widetilde{c}'$ and $\widetilde{c}$ are clear
$\longrightarrow$ invert back.

Descartes' Theorem

Given four mutually tangent circles whose curvatures are a_1, a_2, a_3, a_4 (with the sign convention), then

$$F(a_1, a_2, a_3, a_4) = 0,$$

where F is the quadratic form

$$F(a) = 2(a_1^2 + a_2^2 + a_3^2 + a_4^2) - (a_1 + a_2 + a_3 + a_4)^2.$$

I don't know of the proof "from the book". (If time permits, proof at end.)

Diophantine Property:

Given c_1, c_2, c_3, c_4 mutually tangent circles, a_1, a_2, a_3, a_4 curvatures. If c and c' are tangent to c_1, c_2, c_3, then

$$F(a_1, a_2, a_3, a_4) = 0$$
$$F(a_1, a_2, a_3, a_4') = 0$$

So a_4 and a_4' are roots of the same quadratic equation $\implies$

$$a_4 + a_4' = 2a_1 + 2a_2 + 2a_3 \tag{1}$$

$$a_4, a_4' = a_1 + a_2 + a_3 \pm 2\sqrt{\Delta}$$

$$\Delta = a_1 a_2 + a_1 a_3 + a_2 a_3$$

(our example $(a_1, a_2, a_3) = (21, 24, 28)$, $\Delta = 1764 = 42^2$)
If c_1, c_2, c_3, c_4 have integral curvatures, then c_4' also does from (1)!
In this way, every curvature built is integral.

Apollonian Group:
(1) above $\implies$ that in forming a new curvature when inserting a new circle

$$a_4' = -a_4 + 2a_1 + 2a_2 + 2a_3$$

Set

$$S_4 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad S_3 = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \qquad S_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

$$a' = aS_4, \qquad a' = (a_1, a_2, a_3, a_4') \in \mathbb{Z}^4$$

Similarly with generating $c_1, c_2, c_4, \ldots$.

$$S_j^2 = I, \quad S_j \in \mathrm{GL}_4(\mathbb{Z}), \quad j = 1, 2, 3, 4.$$

Definition

A is the subgroup of $\mathrm{GL}_4(\mathbb{Z})$ generated by S_1, S_2, S_3, S_4; called the Apollonian group.

- It is the symmetry group for any integral Apollonian packing.

If $a \in \mathbb{Z}^4$ is a fourtuple of curvatures of 4 mutually tangent circles in P, then the orbit

$$\mathcal{O}_a = aA \subset \mathbb{Z}^4$$

gives all such 4-tuples in P.

Any a as above satisfies

$$F(a) = 0 \qquad \text{(i.e. we are on a cone)}$$

Not surprisingly,

$$F(xS_j) = F(x)$$

F as a real quadratic form has signature $3,1$ and S_j and hence A are all orthogonal!

O_F the orthogonal group of F.
$\mathrm{O}_F(\mathbb{Z})$ the orthogonal matrices whose entries are integers.

$$A \leq \mathrm{O}_F(\mathbb{Z}).$$

Key feature: (defines our problem)

(i) A is "thin"; it is of infinite index in $\mathrm{O}_F(\mathbb{Z})$

(ii) A is not too small — it is "Zariski dense" in O_F.

The group $\mathrm{O}_F(\mathbb{Z})$ is an arithmetic group. It appears in the modern theory of integral quadratic equations.

- Hilbert's 11-th problem concerns solvability of such equations — solved only recently (2000).

To put Hilbert's 11th problem in context: it is a generalization of the following classical result:
Which numbers are sums of three squares?

$$n = x^2 + y^2 + z^2, \quad x, y, z \in \mathbb{Z}.$$

"Local obstruction": if

$$n = 4^a(8b + 7)$$

then n is not a sum of three squares (consider arithmetic on dividing by 8).

- Gauss/Legendre (1800) (local to global principle): n is a sum of three squares iff $n \neq 4^a(8b + 7)$.

$$V = \{x : F(x) = 0\} \qquad \text{cone in } \mathbb{R}^4$$

$V^{\text{prim}}(\mathbb{Z})$ = points with integer coordinates and $\gcd(a_1, a_2, a_3, a_4) = 1$.

Then

$$V^{\text{prim}}(\mathbb{Z}) = a\mathrm{O}_F(\mathbb{Z})$$

(i.e. one orbit for all points)
$V^{\text{prim}}(\mathbb{Z})$ has infinitely many orbits under A — each corresponding to a different Apollonian packing.

Such "thin groups" come up in many places in number theory. While the powerful modern theory of automorphic forms says nothing about them, there is a flourishing theory of thin groups. It allows for the solution of many related problems.

Counting: $x \geq 1$,

$$N_P(x) := |\{c \in P : a(c) \leq x\}|$$

Theorem (D. Boyd)

$$\lim_{x \to \infty} \frac{\log N_P(x)}{\log x} = \delta = 1.305\ldots$$

"Hausdorff dimension of limit set of the Apollonian Gasket" — elementary arguments

Using tools from hyperbolic 3 manifolds — Laplacians

Theorem (A. Kontorovich–H. Oh 2009)

There is $b = b(P) > 0$ such that

$$N_P(x) \sim bx^{\delta} \quad \text{as } x \to \infty.$$

$$b(P_o) \approx 0.0458\ldots$$

$b(P)$ is determined in terms of the base eigenfunction of the infinite volume "drum" $A \setminus \mathrm{O}_F(\mathbb{R})$.

Diophantine Analysis of P:

Which integers occur as curvatures?

- There are congruence restrictions — that is, in arithmetic on dividing by q, "mod q".

For P_0 for example,

$$\text{every } a(c) \equiv 0, 4, 12, 13, 16, 21 \pmod{24}$$

Theorem (E. Fuchs 2010)

The above is the only congruence obstruction for P_0.

One can examine the reduction

$$A \longrightarrow \mathrm{GL}_4(\mathbb{Z}/q\mathbb{Z}) \quad \text{for } q \geq 1 \text{ (finite group).}$$

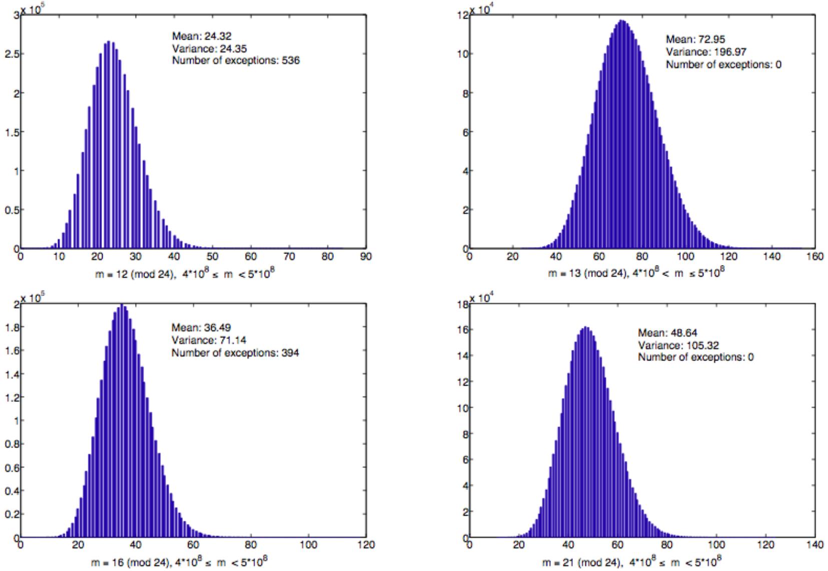
Fuchs determines the precise image.
[Here the "Zariski density" is used; Weisfeiller's work.]

$\#$ of integers $\leq x$ that are hit with multiplicity is $\sim x^{1.3\cdots}$.
So we might hope that a positive density (proportion) of integers are curvatures.

Theorem (J. Bourgain, E. Fuchs 2010)

There is $C > 0$ such that the number of integers $< x$ which are curvatures is at least Cx.

Much more ambitious is the local to global principle: that except for a finite number of integers, every integer satisfying the congruence $\pmod{24}$ is a curvature.


x 10^5
Mean: 24.32
Variance: 24.35
Number of exceptions: 536
m ≡ 12 (mod 24), 4*10^8 ≤ m < 5*10^8
x 10^4
Mean: 72.95
Variance: 196.97
Number of exceptions: 0
m ≡ 13 (mod 24), 4*10^8 < m ≤ 5*10^8
x 10^5
Mean: 36.49
Variance: 71.14
Number of exceptions: 394
m ≡ 16 (mod 24), 4*10^8 ≤ m < 5*10^8
x 10^4
Mean: 48.64
Variance: 105.32
Number of exceptions: 0
m ≡ 21 (mod 24), 4*10^8 ≤ m < 5*10^8

Primes:

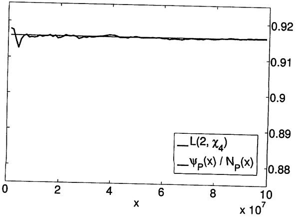
Are there infinitely many prime $a(c)$'s in P_0? Or twins such as 157 and 397 in the middle?

Theorem (S. '07)

In any integral Apollonian packing, there are infinitely many c's with $A(c)$ prime and, better still, infinitely many pairs c, c' with $a(c)$ and $a(c')$ prime.

Is there a prime number theorem? Möbius heuristics suggest yes.

$$\Psi_P(x) := \sum_{\substack{c\in P \\ a(c) \text{ prime} \\ a(c)\le x}} \log a(c), \qquad \frac{\Psi_P(x)}{N_P(x)} \longrightarrow L(2,\chi_4) = 0.9159\ldots??$$



Using the "affine sieve" for thin groups

Theorem (Kontorovich–Oh 2009)

x large,

$$\pi_P(x) = |\{c \in P : a(c) \le x; a(c) \text{ } prime\}| \le \frac{CN_P(x)}{\log x}$$

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